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Nonlinear Heat Transfer in Planar Solids: Direct and Inverse Applications

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For planar geometries, the present investigation develops a method for generating approximate solutions for heat conduction in solids with variable thermal conductivity, for both the direct and inverse problems. In the first portion, the direct case, from theoretical considerations, an analytical solution is generated for the original nonlinear differential system when it is replaced by a sequence of linear differential equations in some optimum fashion. The approach is unlike traditional methods appearing in the literature, since it does not require perturbation parameters and penetration distances. It entails an iterative procedure for accuracy improvement. For the second part of the study, an analytical procedure is developed for the nonlinear inverse problem. The concept of approximating the original material with a pseudolinear one is employed, and thermal diffusivity iteration is introduced. Numerical examples are presented to illustrate the computational procedures.

Nomenclature

a, A_m, b_n, C_n	= general constants or coefficients
B_0	= constant for surface temperature behavior
c	= specific heat
D_n	= constant, Eq. (23)
$i^n \text{erfc} z$	= repeated error integral of variable z
$I(a)$	= mean square difference integral
n,m,q	= number of equal parts or degree of polynomial approximation
P_n	$= \int_0^t (\delta_i t)^n \mathrm{d}t$
t	= time variable
T	= temperature
x	= space variable
z	= similarity variable, Eq. (12)
β	= modified thermal conductivity slope
$\bar{\beta}$	= conventional thermal conductivity slope
δ	= thermal diffusivity
λ	= thermal conductivity
ρ	= density
Δ	= distance between thermocouples, x_2-x_1, or unspecified length

Subscripts

i	= initial value
s	= surface value
n	= nth value
1	= value of first location, or first iteration
2	= value at second location, or second iteration

Introduction

WHEN a solid is either heated or cooled, temperature sensors record the end of the conduction process within the solid. From the physical law governing the thermal response of the solid, a mathematical model representation of this activity may be derived as $(\partial/\partial x)[\lambda(\partial T/\partial x)] = \rho c(\partial T/\partial t)$, the heat conduction equation. If, in turn, the thermal properties are assumed to be constants, then the form of the conduction equation is simplified and is denoted as the linear form. Consequently, two broad categories can be established as they relate to the analysis of the thermal system: the linear or nonlinear conditions. Each classification may be divided further into the two subcategories: direct and inverse. The direct case is referred to when information at the solid's boundaries is used to establish the internal temperature field. The inverse situation is identified when internal responses are employed to predict surface conditions that produced these responses.

For the linear condition, the direct solution of the conventional heat conduction equation is readily available for a wide variety of boundary conditions. Similarly, a solution to the inverse case may be achieved by various methods that appear in the literature. Unfortunately, the nonlinear situation, in general, is intractable, due to the presence of the variable thermal property terms, $\lambda(T)$ and $c(T)$, in the heat conduction equation. Since properties of materials vary with temperature, the local thermal conductivity, $\lambda(T)$ for example, can differ a great deal from its reference value; consequently, the customary replacement of the nonlinear differential equation with its linear approximation will not reveal any information as to the effect of the property variation upon the resultant temperature and heat flux fields. Accordingly, the literature is rich with reported methods for the solution of the nonlinear direct problems. The number of reported studies is considerable; consequently, a limited review is presented for completeness. Its intention is to touch upon representative methods, since more exhaustive listings are readily available in the literature.

To mention a few of the most prominent approaches, Storm[1,2] and Knight and Phillip[3] developed analytical solutions through the use of multiple transformations. Storm's method is successful for semi-infinite domains, and for a particular class of functions for the thermal property variations, i.e., $\lambda(T) \sim e^{aT}$ and λc=constant. In addition, analysis is only possible when the surface heat flux is assumed a constant. Knight demonstrated that Storm's method could be amended to finite solids subjected to adiabatic end conditions. The analytical method requires that $\lambda(T) \sim (a-T)^{-2}$. Pattle[4] presented another closed form solution for the restrictive case of an internal heat source, in an infinite medium whose thermal conductivity follows some power law, i.e., $\lambda(T) \sim T^n$. Extension of the aforementioned methods to other conditions is not possible due to the type of transformations required; therefore, the important situations of time varying boundary conditions and the analysis of finite regions remains to be resolved. Alternatively, perturbation methods have been employed with limited success. The studies of Hopkins,[5] Andre-Talamon,[6] Tsang,[7] Clauson,[8] and

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Olsson[9] are representative of this method of analysis. In the main, these methods are complicated, since the solutions for the higher order perturbation terms are difficult to obtain. Furthermore, the a priori stipulation of "small" may not be representative of a real world condition. For large surface temperature values, the contribution of the perturbation term cannot be regarded as small. The heat balance or integral method, Goodman[10] and Imber and Huang,[11] is another alternative method of analysis. The method is relatively easy to apply; however, the resultant accuracy is limited (see Yang and Szewczyk[12]). In addition, the solution for the penetration distance may entail numerical integration of a complex ordinary differential equation. Yang,[13] and recently, Chung and Yeh[14] and Sugiyama et al.[15] applied this method to multicomponent planar solids. Numerical solutions have appeared in the literature in great numbers. Representative of these approaches are the solutions presented by Yang[16] and Kadambi.[17]

A solution to the linear inverse case may be achieved by several methods that appear in the literature. As shown in Refs. 18 and 19, an analytical procedure is available for the one-dimensional linear inverse problem, when temperature responses are known at two internal locations. Based upon these traces a closed form solution predicts the temperature beyond the two positions. The procedure is also applicable to multicomponent domains. D'Souza,[20] Arledge and Haji-Sheikh,[21] and Hore et al.[22] present numerical methods based upon finite difference or finite-element techniques for obtaining temperature extrapolation results for the linear condition. Chawla et al.[23] and Beck[24] develop alternative numerical methods for computing temperatures or heat fluxes in the nonlinear case, for the direct and inverse cases, respectively. Mehta[25] and Chen and Thomsen[26] indicate additional schemes for temperature computations for their particular interests. Imber[27,28] presents an extrapolation mechanism for heat flow in two dimensions. A listing of the earlier studies is shown in Ref. 18.

In what follows, mechanisms for temperature determination are developed for the direct and inverse situations, respectively, in solids bounded by parallel planes. For the first, an approximate solution is generated based upon the method of equivalent linearization. The key principle in the proposed technique is to replace the original nonlinear partial differential equation with a succession of "equivalent" linear ones in some optimum fashion. In turn, a procedure for estimating extrapolation temperatures for the inverse problem will be demonstrated. Experience gained from application of the aforementioned linearization principles indicates that good results can be achieved for the nonlinear inverse condition when it is treated as a quasilinear problem. Several numerical results are presented as an indication of the method's accuracy.

Nonlinear Direct Case: Semi-Infinite Domain

In most engineering applications, the thermal conductivity variation is regarded as the major contribution in the nonlinear effect. To demonstrate the procedure of equivalent linearization, it will therefore be assumed that $\lambda(T)$ is the only first-order effect. It should be noted that this method can be extended to include variable specific heat; however, the required computations become lengthier. In the following, the principle of equivalent linearization is applied to the equation of linear heat flow for two reasons. First, the solution for planar geometries is of considerable technical interest, and second, the application will provide information on how to treat other configurations. Accordingly, for constant specific heat, the resultant one-dimensional heat conduction equation, Eq. (1)

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) \tag{1}$$

is initially replaced with the tentative linear equation

$$\rho c \frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial x^2} + C_0 \frac{\partial^2 T_0}{\partial x^2} \tag{2}$$

In Eq. (2), a represents a constant parameter to be selected so that the mean square difference of the right-hand side terms in Eqs. (1) and (2) is minimized. In other words, $I(a)$ in Eq. (3) is to be minimized:

$$I(a) = \int_0^\infty \int_0^t \left\{ \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) - a \frac{\partial^2 T}{\partial x^2} - C_0 \frac{\partial^2 T_0}{\partial x^2} \right\}^2 \mathrm{d}x\mathrm{d}t \tag{3}$$

where the limits on the spatial integration coincide with the geometry under consideration. Thus the operation $\mathrm{d}I/\mathrm{d}a = 0$ results in an explicit expression for a:

$$a = \frac{\int_0^\infty \int_0^t \left[\frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) - C_0 \frac{\partial^2 T_0}{\partial x^2} \right] \frac{\partial^2 T}{\partial x^2} \mathrm{d}x\mathrm{d}t}{\int_0^\infty \int_0^t \left(\frac{\partial^2 T}{\partial x^2}\right)^2 \mathrm{d}x\mathrm{d}t} \tag{4}$$

A trial solution is now selected such that it satisfies the boundary conditions and the linear form of Eq. (1). This initial trial is designated as $T_0(x,t)$ and, at this stage in the approximation, it replaces all the temperature terms in Eq. (4). Since the quantity a is prescribed to be a constant, any temporal contributions from the right-hand side of Eq. (4) must be negated. Hence it is envisaged that the parameter C_0 will be adjusted so that it will negate the contributions of the time-dependent integrals in Eq. (4) if necessary; consequently, the aforementioned adjustment produces in a straightforward manner the result that $a = \lambda_i$, the reference thermal conductivity. As a second possibility, complete cancellation of the troublesome integrals can occur in Eq. (4) without prescribing a value for C_0. In this instance, Eq. (4) results in a simple algebraic expression relating the term a to the parameter C_0. Since C_0 is now a free parameter, its value is selected so that the same end result is obtained, $a = \lambda_i$. The advantage in doing so is that this step decouples the determination of the temperature field from the problem of evaluation of mean thermal properties. Hence, the thermal properties are evaluated at the initial temperature and the method can proceed without the bothersome concern as to the development of mean thermal properties. The rationale for the adjustment in C_0 is best illustrated by the numerical examples which follow, and its effect is the same as stipulating that $a = \lambda_i$. In turn, the temperature formulation obtained from Eq. (2) may be regarded as the resultant of a heat generation perturbation, $C_0 \partial^2 T_0 / \partial x^2$, upon the linear condition. Proceeding with the method, Eq. (2) is iterated; therefore, its replacement is now

$$\rho c \frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial x^2} + C_0 \frac{\partial^2 T_0}{\partial x^2} + C_1 \frac{\partial^2 T_1}{\partial x^2} \tag{5}$$

where the term $T_1(x,t)$ denotes the temperature expression obtained from Eq. (2) subject to the original boundary conditions. Accordingly, Eq. (3) must be amended to reflect the presence of the new heat generation term $C_1(\partial^2 T_1/\partial x^2)$. Thus

$$I(a) = \int_0^\infty \int_0^t \left\{ \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) - a \frac{\partial^2 T}{\partial x^2} - C_0 \frac{\partial^2 T_0}{\partial x^2} \right\}^2 \mathrm{d}x\mathrm{d}t \tag{6}$$

and the minimization operation $\mathrm{d}I/\mathrm{d}a = 0$ determines

$$a = \frac{\int_0^\infty \int_0^t \left[\frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) - C_0 \frac{\partial^2 T_0}{\partial x^2} - C_1 \frac{\partial^2 T_1}{\partial x^2} \right] \frac{\partial^2 T}{\partial x^2} \mathrm{d}x\mathrm{d}t}{\int_0^\infty \int_0^t \left(\frac{\partial^2 T}{\partial x^2}\right)^2 \mathrm{d}x\mathrm{d}t} \tag{7}$$

To negate any temporal contributions that arise due to the integration with respect to time in the preceding equation, the constants C_0 and C_1 are chosen accordingly. In effect, a portion of Eq. (7) yields an equation in C_0 and C_1, with the companion requirement that the constant a is always determined to be the reference thermal conductivity λ_i. A second equation for C_0 and C_1 is established by optimization with respect to C_1. Hence C_1 is optimized by the relation $dI(a, C_0, C_1)/dC_1 = 0$. In turn, the approximate solution for the temperature profile is obtained from the iterated equation, Eq. (5). More generally, the "equivalent" equation replacement can be expressed as:

$$\rho c \frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial x^2} + \sum_{n=0}^{N} C_n \frac{\partial^2 T_n}{\partial x^2} \tag{8}$$

along with the optimization requirements of

$$dI/da = 0$$

and

$$dI/dC_n = 0, \qquad n \neq 0 \tag{9}$$

where

$$a = \frac{\int_0^\infty \int_0^t \left[\frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) - \sum_{n=0}^{N} C_n \frac{\partial^2 T_n}{\partial x^2}\right] \frac{\partial^2 T}{\partial x^2} dx dt}{\int_0^\infty \int_0^t \left(\frac{\partial^2 T}{\partial x^2}\right)^2 dx dt} \tag{10}$$

Since the solution for the initial linear equation is always substituted for the nonsubscripted temperature terms appearing in Eq. (1), the evaluation of the constant a presents little difficulty. The requirement that the parameter a be a constant coupled with the effect of Eq. (9) yields sufficient relationships to determine all the constants, C_n. The end result of this process is that $a = \lambda_i$.

It should be noted that the preceding theory requires that the integration operations affiliated with the terms C_n ultimately yield a set of simultaneous equations each of which is spatially and temporally independent. It can be anticipated that simple adjustments in C_n may not be sufficient to achieve this in all instances. Consequently, the form of the term $\partial^2 T_n/\partial x^2$ in the tentative linear equation may have to be adjusted as well to accomplish this. Hence the term $\partial^2 T_n/\partial x^2$ serves as a guide for the selection of the appropriate form to be used. Application of limiting conditions and integration operations will determine the actual form of the modified heat generation quantity. Further details of this procedure appear in the numerical examples that follow.

Numerical Examples

As a first illustration of the method, the approximate solution is obtained for a semi-infinite solid subjected to a constant surface temperature T_s, initially at a temperature T_i. For this nonlinear solid, the local thermal conductivity is assumed to vary linearly with temperature. This example was selected since two solutions have been reported in the literature. Yang's results[6] are achieved by numerical integration of the appropriate differential equation and are henceforth referred to as the exact solution. Vujanovic[29] obtains a solution by an alternative method of equivalent linearization. Consequently, a basis for comparison is available.

Proceeding with the analysis, the thermal conductivity variation is written as

$$\lambda(T) = \lambda_i \left[1 + \beta\left(\frac{T - T_i}{T_s - T_i}\right)\right] \tag{11}$$

where the parameter β can be regarded as the modified slope of the conventional linear representation for the thermal conductivity expression. For the linear differential system under consideration, the trial function is

$$\frac{T_0(x,t) - T_i}{T_s - T_i} = \text{erfc } (z)$$

where

$$z = \frac{x}{2(\delta_i t)^{1/2}} \tag{12}$$

Substituting Eq. (12) into Eqs. (2) and (4), respectively, produces the iterated equation

$$\rho c \frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial x^2} + \frac{C_0 z e^{-z^2}}{\delta_i t} \tag{13}$$

with

$$a = \frac{(32\pi)^{-1/2}\lambda_i (T_s - T_i)^2 P_{-3/2} + 0.0905877\beta\lambda_i (T_s - T_i)^2 P_{-3/2}}{(32\pi^{-1/2}(T_s - T_i)^2 P_{-3/2}} + \frac{-(32)^{-1/2} C_0 (T_s - T_i) P_{-3/2}}{(32\pi)^{-1/2}(T_s - T_i)^2 P_{-3/2}}$$

and

$$P_{-n} = \int_0^t (\delta_i t)^{-n} dt \tag{14}$$

The parameter C_0 is now selected so that the last two terms cancel one another; consequently,

$$a = \lambda_i$$

and

$$C_0 = 0.9082789\pi^{-1/2}\beta\lambda_i (T_s - T_i) \tag{15}$$

Hence, the first iterated solution is obtained from Eq. (2) and is

$$\frac{T_1(x,t) - T_i}{T_s - T_i} = \text{erfc } (z) + 0.5124415\,\beta z e^{-z^2} \tag{16}$$

If the heat generation terms associated with Eq. (5) are now formed, then the temperature behavior is described by the new differential equation

$$\rho c \frac{\partial T}{\partial t} = a \frac{\partial^2 t}{\partial x^2} + \frac{C_0 z e^{-z^2}}{\delta_i t} + \frac{C_i z^3 e^{-z^2}}{\delta_i t} \tag{17}$$

It should be noted that the operations $\partial^2 T_0/\partial x^2$ and $\partial^2 T_1/\partial x^2$ produce some terms that are of the same form; consequently, it is expeditious to group these terms together, and the resultant is Eq. (17). Accordingly, from Eq. (7)

$$\begin{aligned} a = {} & [(32\pi)^{-1/2}\lambda_i (T_s - T_i)^2 P_{-3/2} + 0.0905877\beta\lambda_i \\ & \times (T_s - T_i)^2 P_{-3/2}] / [(32\pi)^{-1/2}(T_s - T_i)^2 P_{3/2}] \\ & + [-(32)^{-1/2} C_0 (T_s - T_i) P_{-3/2} - (18/1024)^{1/2} C_1 \\ & \times (T_s - T_i)^2 P_{-3/2}] / [(32\pi)^{-1/2}(T_s - T_i)^2 P_{-3/2}] \end{aligned} \tag{18}$$

From the second optimizing relationship $dI/dC_1 = 0$, the following equation is obtained after some algebraic manipulation

$$3.2\, C_0 + 4C_1 = 1.0678997\,\beta\lambda_i (T_s - T_i) \tag{19}$$

The parameters C_0 and C_1 are now selected so that $a = \lambda_i$, hence from Eq. (18)

$$4C_0 + 3C_1 = 2.0497659\,\beta\lambda_i(T_s - T_i) \tag{20}$$

The terms C_0 and C_1 can be explicitly determined from the preceding relationships; consequently, the resultant temperature after two iterations, is

$$\frac{T_2(x,t) - T_i}{T_s - T_i} = \text{erfc}\ (z) + \{0.5124415 - 0.1787228z^2\}\beta z e^{-z^2} \tag{21}$$

which is the solution to Eq. (17). Further iterations are produced in a similar manner as previously described, where the optimizations are $dI/dC_n = 0$. It can be anticipated that the associated differential equation will be of the form

$$\rho c \frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial x^2} + \frac{1}{\delta_i t} \sum_{n=0}^{N-1} C_n z^{2n+1} e^{-z^2} \tag{22}$$

and the temperature expression of the general form

$$\frac{T_n(x,t) - T_i}{T_s - T_i} = \text{erfc}(z) + \sum_{n=0}^{N-1} D_n z^{2n+1} \beta e^{-z^2} \tag{23}$$

where the quantity D_n is a constant to be determined by application of the preceding theoretical principles.

A comparison of the computed dimensionless temperatures for the three methods is shown in Table 1 for the different β values, $\beta = \pm(0.4)$ and $\beta = \pm(0.5)$, respectively. For these values of β, numerical values were computed in accordance with the approaches developed in the cited references. In the region of significant temperature gradients, the results indicated that the proposed method of equivalent linearization compares favorably with the exact method. Because the temperature response curve is relatively flat for large values of the dimensionless variable z, improvements in accuracy in this range can be effected by performing additional iterations. In this instance, Eq. (21) would be modified so that more terms would appear in the bracketed term on the right-hand side of the equation, as per Eq. (23). Since the numerical values for the tabulated dimensionless temperatures are already quite small, additional iterations were not pursued. For smaller and more realistic values of β, computations reveal that the differences between the exact and iterated procedure are reduced considerably throughout the entire range in the parameter z.

In conclusion, for the numerical example just presented, by rather simple algebraic manipulations, the constant C_n can be selected so that $a = \lambda_i$, and Eq. (21) is obtained in a straightforward manner from the indicated equations. Because of the constant surface temperature boundary condition, no adjustments in the form of the terms generated by the tentative heat contributions $C_n(\partial^2 T_n/\partial x^2)$ were attempted. In other words, the form of these terms is compatible with the requirements that: $a = \lambda_i$, the iterated temperature solution is easily generated, and the limiting conditions with respect to, x, t, and β are satisfied. Inspection of Eqs. (1) and (17) also reveals that, in close proximity of the wall, a small error is incurred due to the fact that the actual thermal conductivity λ is not at λ_s. Computations reveal that the magnitude of this quantity reduces rapidly with decreasing values of the nonlinearity parameter β. Outside of this local region, Eq. (21) corresponds much more closely to the exact solution than does the method of Ref. 29. As demonstrated by the next example, the tentative heat generation terms will not always produce expressions which are compatible with the aforementioned requirements; hence some adjustments must be made.

For the second illustration of the method, an approximate solution is generated for a semi-infinite solid initially at T_i, subjected to a time varying boundary condition, i.e., $T_s - T_i = B_0\sqrt{t}$ where B_0 is an arbitrary constant. Hence the trial function is

$$T_0(x,t) - T_i = B_0\Gamma(3/2)(4t)^{1/2} i\ \text{erfc}\ (z) \tag{24}$$

since it satisfies the linear form of Eq. (1) and the related boundary conditions. For substitution in Eq. (2), the differentiation of the trial function $T_0(x,t)$ results in the suggested form $B_0 e^{-z^2}/2\delta_i t^{1/2}$. The form of this contribution is inappropriate since substitution of this terms into Eq. (4) will not produce an expression for a which is temporally independent. At this step in the analysis, the pseudo-heat generation term is amended in form so that it is now $ze^{-z^2} + \text{erfc}\ (z)$; consequently, the preceding reservation is eliminated. The first term follows from the solution method

Table 1 Comparison of results for semi-infinite solid

z	Eq. (21)	Ref. (29)	Exact	Eq. (21)	Ref. (29)	Exact
		$\beta = -0.4$			$\beta = 0.4$	
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	0.8673	0.8594	0.8465	0.9078	0.9036	0.9088
0.2	0.7385	0.7230	0.7101	0.8161	0.8086	0.8165
0.4	0.5056	0.4784	0.4852	0.6376	0.6286	0.6339
0.6	0.3211	0.2876	0.3178	0.4712	0.4674	0.4639
0.8	0.1907	0.1562	0.1985	0.3251	0.3326	0.3173
1.0	0.1082	0.0763	0.1176	0.2064	0.2258	0.2016
1.2	0.0607	0.0334	0.0658	0.1187	0.1461	0.1184
1.4	0.0349	0.0131	0.0346	0.0605	0.0899	0.0643
1.6	0.0209	4.58×10^{-3}	0.0170	0.0264	0.0526	0.0322
1.8	0.0128	1.42×10^{-3}	7.83×10^{-3}	9.03×10^{-3}	0.0292	0.0150
2.0	7.64×10^{-3}	3.95×10^{-4}	3.35×10^{-3}	1.71×10^{-3}	0.0154	6.44×10^{-3}
		$\beta = -0.5$			$\beta = 0.5$	
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	0.8623	0.8482	0.8297	0.9128	0.9066	0.9126
0.2	0.7287	0.7019	0.6858	0.8258	0.8146	0.8238
0.4	0.4891	0.4439	0.4594	0.6541	0.6390	0.6465
0.6	0.3024	0.2508	0.2972	0.4899	0.4816	0.4789
0.8	0.1739	0.1257	0.1842	0.3419	0.3481	0.3317
1.0	0.0959	0.0556	0.1086	0.2187	0.2409	0.2131
1.2	0.0534	0.0216	0.0606	0.1259	0.1593	0.1263
1.4	0.0317	7.37×10^{-3}	0.0318	0.0637	0.1006	0.0690
1.6	0.0203	2.19×10^{-3}	0.0156	0.0270	0.0606	0.0348
1.8	0.0133	5.7×10^{-4}	7.19×10^{-3}	8.56×10^{-3}	0.0348	0.0162
2.0	8.39×10^{-3}	1.3×10^{-4}	3.08×10^{-3}	9.70×10^{-4}	0.0190	6.97×10^{-3}

in the first example and the second term is an attempt to improve the accuracy in the vicinity near the surface as simply as possible. In addition, the second term is included since all subsequent iterations are generically obtained from a term of this form. Thus, for the first iteration,

$$\rho c \frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial x^2} + C_0 \{ z e^{-z^2} + \text{erfc}\ (z) \} \tag{25}$$

and from Eq. (4), for $a = \lambda_i$

$$C_0 \{ 1 + \pi^{1/2} \} = 1.5768178 \lambda_i \bar{\beta} B_0^2 / \delta_i \tag{26}$$

where the conventional conductivity slope $\bar{\beta}$ is obtained from

$$\lambda(T) = \lambda_i \{ 1 + \bar{\beta}(T - T_i) \} \tag{27}$$

Hence the first iterated solution follows from Eq. (25),

$$T_1(x,t) - T_i = B_0 (\pi t)^{1/2} \{ 1 + \pi^{1/2}$$
$$+ 0.7884089 \bar{\beta} B_0 t^{1/2} z / 1 + \pi^{1/2} \} i\text{erfc}(z)$$
$$+ \{ 1.3974184 \bar{\beta} B_0^2 t z^2 / 1 + \pi^{1/2} \} \text{erfc}(z)$$
$$+ \{ 3.1536356 \bar{\beta} B_0^2 t z / 1 + \pi^{1/2} \} i\text{erfc}(z) \tag{28}$$

For continuation of the iteration procedure, the pseudo-heat generation term $\partial^2 T_1 / \partial x^2$ is first formed. The resultant differentiation, in turn, produces the term $e^{-z^2} / t^{1/2}$, which is replaced by the amended term $z e^{-z^2}$ since it satisfies all the previously mentioned prerequisites. Iterations are continued in a similar manner and the relevant equations for four iterations are

$$\rho c \frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial x^2} + C_0 \{ z e^{-z^2} + \text{erfc}(z) \}$$
$$+ C_1 z e^{-z^2} + 8 C_2 z^3 e^{-z^2} + 32 C_3 z^5 e^{-z^2} \tag{29}$$

and

$$2.7724539\ C_0 + C_1 + 4C_2 + 16C_3 = 1.5768178 \lambda_i \bar{\beta} B_0{}^2 / \delta_i \tag{30a}$$

$$1.9347799\ C_0 + C_1 + 6C_2 + 30C_3 = 0.6096144 \lambda_i \bar{\beta} B_0{}^2 / \delta_i \tag{30b}$$

$$0.2375439\ C_0 + 0.1C_1 + C_2 + 7C_3 = 0.0185186 \lambda_i \bar{\beta} B_0{}^2 / \delta_i \tag{30c}$$

$$0.0154052\ C_0 + 7.9366 \times 10^{-3} C_1 + 0.1111 C_2 + C_3$$
$$= 6.04812 \times 10^{-4} \lambda_i \bar{\beta} B_0{}^2 / \delta_i \tag{30d}$$

where the simultaneous equations are obtained from the optimization condition $dI/dC_n = 0$, $n \neq 0$, and $a = \lambda_i$. For brevity these details are not shown since they entail simple algebraic manipulations. The resultant temperature after four iterations is therefore

$$T_4(x,t) - T_i = (\pi t)^{1/2} \{ B_0 + (C_0 + C_1) z t^{1/2} / 2 \} i\ \text{erfc}\ (z)$$
$$+ \{ (C_0 + C_1) \pi^{1/2} t z^2 / 2 \}\ \text{erfc}\ (z) + 2 C_0 z t\ i\ \text{erfc}(z)$$
$$+ \{ 8 (C_2 + 5C_3) z^3 t / 3 \} e^{-z^2} + 2 (C_2 + 5C_3) z t e^{-z^2}$$
$$+ 8 C_3 z^5 t e^{-z^2} \tag{31}$$

Before proceeding to a numerical demonstration, it should be noted that the contribution $z e^{-z^2} - \text{erfc}(z)$ is equally valid for the first iteration. In this instance, Eq. (25) is simply amended by replacement of the positive sign in the last term by a minus quantity. Hence Eq. (31) can be regarded as the general solution, where the constants C_n are determined from:

$$-0.7724533 C_0 + C_1 + 4C_2 + 16C_3 = 1.5768178 \lambda_i \bar{\beta} B_0{}^2 / \delta_i \tag{32a}$$

$$-0.0652201 C_0 + C_1 + 6C_2 + 30C_3 = 0.6096144 \lambda_i \bar{\beta} B_0{}^2 / \delta_i \tag{32b}$$

$$-0.0375439 C_0 + 0.1 C_1 + C_2 + 7C_3 = 0.0185186 \lambda_i \bar{\beta} B_0{}^2 / \delta_i \tag{32c}$$

$$4.67972 \times 10^{-4} C_0 + 7.9366 \times 10^{-3} C_1 + 0.1111 C_2 + C_3$$
$$= 6.04812 \times 10^{-4} \lambda_i \bar{\beta} B_0{}^2 / \delta_i \tag{32d}$$

In the example that follows, the mean square difference integral $I(a)$ from Eq. (6) was evaluated for the two solutions based upon the terms $z e^{-z^2} \pm \text{erfc}(z)$. For negative values of the parameter $\bar{\beta}$ the resultant integral $I(a)$ was less in value when the term $z e^{-z^2} + \text{erfc}(z)$ was considered as the replacement. Positive values of $\bar{\beta}$ produced a smaller value for the integral $I(a)$ when $z e^{-z^2} - \text{erfc}(z)$ was utilized; consequently, the solution may be regarded as more accurate.

Accordingly, for a semi-infinite solid whose surface temperature follows the relationship $T(0,t) - T_i = t^{1/2}$, numerical solutions were obtained for different values in the thermal conductivity parameter $\bar{\beta}$. A comparison of these values with the presented iteration method is shown in Table 2 where, for the numerical computation, the thermal diffusivity

Table 2 Semi-infinite solid with $T(0, t) - T_i = t^{1/2}$

	$t = 0.5$ h				$t = 1$ h			
	$\bar{\beta} = -0.5$		$\bar{\beta} = 0.5$		$\bar{\beta} = -0.5$		$\bar{\beta} = 0.5$	
x, ft	Eqs. (30) and (31)	Exact	Eqs. (31) and (32)	Exact	Eqs. (30) and (31)	Exact	Eqs. (31) and (32)	Exact
0.00	0.70711	0.70711	0.70711	0.70711	1.00000	1.00000	1.00000	1.00000
0.25	0.58991	0.58088	0.61412	0.62092	0.86937	0.85192	0.90874	0.91789
0.50	0.48052	0.47752	0.53127	0.53992	0.74982	0.73107	0.82523	0.83854
0.75	0.38918	0.39179	0.45687	0.46464	0.64181	0.62969	0.74816	0.76224
1.00	0.31504	0.32026	0.38503	0.39550	0.54676	0.54326	0.67659	0.68965
1.25	0.24926	0.26047	0.32800	0.33283	0.46134	0.46877	0.60434	0.61984
1.50	0.19927	0.21057	0.27173	0.27678	0.38868	0.40417	0.54561	0.55422
1.75	0.16052	0.16909	0.22054	0.22738	0.32729	0.34799	0.48498	0.49261
2.00	0.13106	0.13474	0.17454	0.18448	0.27632	0.29893	0.42703	0.43517
2.50	0.09172	0.08343	0.09944	0.11692	0.20173	0.21894	0.31964	0.33160
3.00	0.06662	0.04975	0.04808	0.07044	0.15537	0.15828	0.22476	0.24843
3.50	0.04464	0.02851	0.02099	0.04035	0.12674	0.11258	0.14555	0.18032
4.00	0.02382	0.01569	0.01149	0.02199	0.10644	0.07860	0.08476	0.12737

δ_i is specified as $1.12093\times10^{-4}\text{m}^2/\text{s}$. The tabulated results indicate that good agreement is achieved in the significant temperature gradient region. As previously mentioned, further iterations can be performed, resulting in additional improvements for large values in the variable x. It should be noted that the values of $\bar{\beta}=\pm0.5$ constitutes an extreme condition. For engineering applications, the values of $\bar{\beta}$ are much less, and closer agreement can be anticipated.

With the presentation of the previous example, the phase of investigation concerned with temperature determination for the nonlinear direct case is concluded. In summary, Iwan[30,31] has applied the conventional equivalency concepts to dissipative systems modeled by ordinary second-order nonlinear differential equations. Based upon the a priori requirement of periodicity, approximate solutions are generated. It should be noted that the current version of linearization goes further insofar as it removes any stipulation of periodicity, and it is applicable to partial differential equations. Since the analysis incorporates an iteration feature, a mechanism for accuracy improvement is available. Vujanovic[29] develops an approximate solution for the first numerical example, described earlier. Because of the optimization approach used, the method is limited since it cannot be extended to include time varying boundary conditions. The current analysis does not have this restriction.

As a first demonstration of the principle of equivalent linearization, solutions are generated for heat conduction systems described by Eq. (1). For other geometrical configurations such as cylinders or spheres, the method can be applied similarly, since all that is required is the appropriate expression for the heat generation term. However, it can be envisaged that solution to the iterated differential equation will be more difficult to effect due to the presence of Bessel functions if, for example, radial heat flow were under consideration. Furthermore, the required integrations indicated in Eq. (4) may be difficult to perform analytically, and numerical approaches would have to be adopted.

Nonlinear Inverse Case

In the nonlinear situation, temperature prediction from embedded thermal sensors can be envisaged as difficult since there is a coupling between the local thermal conductivity and temperature which is not present in the linear case. In the absence of any analytical theory for the solution of the direct case, can a suitable extrapolation method be developed which would estimate temperatures well?

As shown in Refs. 18 and 19, an analytical procedure is available for the one-dimensional linear inverse problem, when temperature responses are known at two internal locations. Based upon these traces, a closed form solution predicts the temperature beyond the two positions. For the nonlinear application, the principles of equivalent linearization can be incorporated into the aforementioned linear analysis; however, the resultant equations become so unwieldy that their usefulness is limited. Alternatively, it should be noted that in the direct case it was observed that portions of the temperature response curve could be closely approximated by a linear material whose thermal diffusivity was adjusted. In other words, the exact solution may be approximated by a linear solution with restricted range, if the original diffusivity is replaced by an equivalent value. In the direct case, sufficient information is not available to determine this replacement. Fortuitously, the inverse problem has this capability. Accordingly, in what follows, the treatment of the nonlinear material as a pseudolinear material will be pursued. The strategy is to position a third thermocouple halfway between the two internal locations previously cited. Then by comparing the linear output with the sensor result, a numerical adjustment can be effected whenever necessary so that the two outputs are brought closer together. In principle, linear extrapolation may be applied for the appropriate time interval, since the heat conduction equation can now be regarded as pseudolinear. The inverse problem also has an

Table 3 Comparison of midpoint temperatures

t, h	% Error $n,m=6$	% Error $n=11$, $m=10$
0.1	0.438	−0.035
0.2	−7.191	−0.099
0.3	−0.147	−0.036
0.4	−2.056	−0.069
0.5	−0.008	0.009
0.6	−1.438	−0.069
0.7	−0.052	0.018
0.8	1.902	−0.081
0.9	0.065	0.017
1.0	−5.591	−0.075
1.1	−0.409	−0.063

Table 4 Surface prediction temperatures for $\bar{\beta}=-0.15$

t, h	Exact	Eq. (33) $n,m=6$	Eq. (33) $n=11$, $m=10$
0.1	1.0	1.0133	0.9996
0.2	1.0	1.0886	0.9976
0.3	1.0	0.9958	0.9988
0.4	1.0	0.9732	0.9991
0.5	1.0	1.0008	0.9988
0.6	1.0	1.0166	1.0000
0.7	1.0	0.9979	0.9996
0.8	1.0	0.9758	0.9997
0.9	1.0	1.0022	0.9999
1.0	1.0	1.0693	0.9989
1.1	1.0	0.9874	1.0002

Table 5 Predicted temperature from linear theory

t, h	% Error $x=0.0762$ m a	b	$x=0$ m a	b	$x=0.0286$ m a	b	$x=0.0190$ m a	b	$x=0.0095$ m a	b
0.1	−3.44	−2.29	11.00	9.36	1.88	1.48	4.30	3.50	7.32	6.10
0.2	0.03	−3.50	11.86	8.38	5.77	1.87	7.62	3.80	9.66	5.98
0.3	−2.23	−2.88	5.40	6.35	1.07	1.15	2.33	2.67	3.77	4.41
0.4	−3.12	−2.10	4.14	5.35	−0.16	1.13	1.08	2.36	2.52	3.77
0.5	−1.94	−1.79	4.88	4.43	0.97	0.86	2.11	1.90	3.42	3.09
0.6	−1.06	−1.52	4.58	3.94	1.92	0.78	2.34	1.71	3.40	2.75
0.7	−1.23	−1.19	3.16	3.34	0.62	0.73	1.36	1.44	2.20	2.38
0.8	−1.67	−0.74	2.30	2.73	−0.11	0.42	0.58	1.21	1.38	2.08
0.9	−1.23	0.29	3.36	2.24	0.65	1.05	1.43	1.37	2.33	2.12
1.0	0.14	−2.58	4.93	2.91	2.33	−2.00	3.10	0.54	3.97	0.49
1.1	−0.32	4.94	0.22	−0.01	0.19	0.13	0.88	2.99	0.23	−0.10

Note: Column a is for $n=m=6$; column b is for $n=9$, $m=8$.

additional favorable feature in that the entire spatial range does not have to be considered; only a portion of it needs to be considered. Extrapolation is therefore carried out over small distances from the first positioned thermal sensor, where the distance Δ between the thermal sensors is a modest value.

For completeness, the expression for the prediction temperature and the companion equations are shown as follows:

$$T(x,t) = \sum_{m=1}^{M} A_m \sum_{n=1}^{N} b_n \sum_{q=0,2,4}^{\infty} (4t)^{n_l 2n} \times \left\{ \operatorname{erfc}\frac{x-x_l(m+q)\Delta}{2(\delta t)^{1/2}} - \operatorname{erfc}\frac{x_l-x+(m+q+2)\Delta}{2(\delta t)^{1/2}} \right\} + \sum_{n=1}^{N} b_n \sum_{q=1,3,5}^{\infty} (4t)^{n_l 2n} \times \left\{ \operatorname{erfc}\frac{x_l-x+q\Delta}{2(\delta t)^{1/2}} - \operatorname{erfc}\frac{x-x_l+q\Delta}{2(\delta t)^{1/2}} \right\} \quad \Delta \geq x_l \tag{33}$$

with

$$T_l(t) = \sum_{m=1}^{M} A_m \sum_{n=1}^{N} b_n (4t)^{n_l 2n} \operatorname{erfc}\frac{m\Delta}{2(\delta t)^{1/2}} \tag{34}$$

and

$$T_2(t) = \sum_{n=1}^{N} b_n t^n / n! \tag{35}$$

The details of the derivation of these equations may be found in Ref. 18. For the application under consideration, the thermal diffusivity value is $\delta = \delta_i$, which is the initial or reference state point which is to be amended.

Numerical Examples

For the following, the inversion method is applied to a semi-infinite solid initially at zero temperature and whose surface is raised to unity. From the numerical solution to the direct nonlinear problem, the temperature traces at the interior spatial positions x_l and x_2 are generated. For backward extrapolation, the thermocouple reponse at x_2 is approximated by an nth degree polynomial over a time range which has been subdivided into N equal parts; consequently, the coefficients b_n are determined by least squares and orthogonal polynomial concepts. In turn, the discretized temperature data at x_l are substituted into Eq. (34) and the values of the A_m coefficients are established. Equation (33) is then applied to predict the temperature at the desired location.

In an earlier paper[18] linear extrapolation was performed on a semi-infinite solid composed of copper with $\delta_i = 1.12093 \times 10^{-4} \mathrm{m}^2/\mathrm{s}$ and with thermocouples positioned at $x_l = 0.0381$ m and $x_2 = 0.1143$ m, respectively. For the nonlinear demonstration, all properties and positions were kept the same; however, the nonlinear parameter β was selected so that $\beta = 0.15$. From published thermal conductivity data on copper, this value represents an upper bound on β for a real world simulation.

As already indicated in the text, in addition to the two conventional thermocouples, a third is arbitrarily introduced which is halfway between the two. The linear theory is then used as interpolation to the midposition for comparison with the actual thermal trace. The tabulated results in Table 3 indicate that good agreement is achieved; consequently, the thermal behavior is approximated closely by the linear theory over the spatial region under consideration. The actual extrapolate values to the surface are indicated in Table 4.

As shown in Tables 3 and 4, the treatment of the nonlinear response as a linear one is very successful. Increasing the degree of the polynomials in n and m results in improved accuracy. The pair, $n = 11$, $m = 10$ was chosen since the combination resulted in the best approximation for the temperature at $x = x_l$. In addition, comparison of the computed and actual traces at $x = x_l + \Delta/2$ indicated that the results were sufficiently close enough so that no adjustment in δ_i was required. Obviously, the numerical values of the coefficients b_n and A_m for the nonlinear conditions are different from those in Ref. 18; however, extrapolation in both instances is successful. Thus for materials which exhibit weak nonlinear behavior, it can be anticipated that good results may be obtained from the straightforward application of the linear extrapolation mechanism.

For materials considered to be strongly nonlinear, application of unamended linear inverse theory will not produce

Table 6 Predicted temperature for δ iteration

t, h	% Error $x = 0.0762$ m, a	b	$x = 0$ m, a	b
0.1	−0.004	0.06	8.95	7.80
0.2	0.07	−3.05	11.07	7.88
0.3	−1.78	−2.65	4.78	6.07
0.4	−2.89	−1.96	3.89	5.16
0.5	−1.82	−1.68	4.75	4.29
0.6	−0.95	−1.45	4.41	3.84
0.7	−1.12	−1.28	2.96	3.35
0.8	−1.58	−1.21	2.18	3.02
0.9	−1.19	−0.82	3.36	2.99
1.0	0.15	2.03	4.93	−4.71
1.1	−0.19	−1.26	−0.28	4.74

Note: Column a is for $n = m = 6$; effective diffusivity is 1.06512×10^{-5} m^2/s; column b is for $n = 9$, $m = 8$; effective diffusivity is 1.12851×10^{-5} m^2/s.

Table 7 Predicted temperature from linear theory

$\Delta = 0.0381$ m $x_l = 0.0190$ m $x_2 = 0.0571$ m

% Error

t, h	$x = 0.0381$ m, a	b	$x = 0$ m, a	b	$x = 0.0143$ m, a	b	$x = 0.0095$ m, a	b	$x = 0.0048$ m, a	b
0.1	−0.95	−1.82	2.40	4.52	0.45	0.89	1.00	1.94	1.64	3.15
0.2	−6.65	−1.43	−4.08	2.97	−6.42	0.39	−5.81	1.13	−5.02	1.99
0.3	−0.98	−0.75	2.68	2.32	0.52	0.52	1.14	1.04	1.87	1.64
0.4	0.73	−0.69	3.64	1.71	2.10	0.28	2.56	0.69	3.07	1.17
0.5	−0.47	−0.50	1.29	1.53	0.25	0.33	0.55	0.67	0.90	1.07
0.6	−1.31	−0.44	0.09	1.27	−0.84	0.26	−0.58	0.55	−0.27	0.81
0.7	−0.45	−0.40	1.27	1.10	0.24	0.21	0.54	0.46	0.88	0.76
0.8	0.77	−0.30	2.52	1.03	1.57	0.24	1.85	0.47	2.17	0.73
0.9	−0.21	−0.43	0.57	0.73	0.11	0.05	0.24	0.24	0.40	0.46
1.0	−3.40	−0.22	−3.69	1.07	−3.79	0.38	−3.78	0.61	−3.75	0.78
1.1	−0.73	−0.74	2.24	−0.36	0.41	0.07	0.93	0.21	1.54	0.08

Note: Column a is for $n = m = 6$; column b is for $n = 9$, $m = 8$.

Table 8 Predicted temperature for δ iteration

t, h	% Error			
	$x=0.0381$ m		$x=0$ m	
	a	b	a	b
0.1	−0.08	0.07	0.84	0.74
0.2	−6.57	−3.36	−4.08	−0.09
0.3	−0.97	−0.28	2.72	3.05
0.4	−0.78	−0.27	3.54	1.91
0.5	−0.41	−0.77	1.17	0.96
0.6	−1.28	−0.44	0.05	1.29
0.7	−0.45	−0.09	1.29	1.40
0.8	0.77	−0.50	2.50	0.53
0.9	−0.16	−0.69	0.43	0.49
1.0	−3.35	0.92	−3.82	2.89
1.1	−0.87	0.21	2.76	−0.80

Note: Column a is for $n=m=6$; effective diffusivity is 0.93981×10^{-5} m^2/2. column b is for $n=m=7$; effective diffusivity is 0.85291×10^{-5} m^2/2.

Table 9 Surface prediction temperature for $\beta=-0.5$

t, h	Exact	$n, m=6$	$n, m=7$	$n=9, m=8$
0.1	1.0	0.9931	0.9917	0.9895
0.2	1.0	1.0803	1.0329	1.0012
0.3	1.0	0.9941	0.9843	0.9922
0.4	1.0	0.9690	0.9907	0.9967
0.5	1.0	0.9976	1.0040	0.9954
0.6	1.0	1.0164	0.9973	0.9966
0.7	1.0	0.9973	0.9912	0.9977
0.8	1.0	0.9721	1.0040	0.9966
0.9	1.0	0.9994	1.0075	1.0011
1.0	1.0	1.0756	0.9624	0.9932
1.1	1.0	0.9929	1.0024	1.0019

results that are as accurate. This is especially true if the extrapolation distance covered is large. To illustrate this, a numerical example is presented for mild steel where $\delta_i=1.276556\times10^{-5}\text{m}^2/\text{s}$ and $\beta=-0.5$. For purposes of comparison with the preceding example, the thermocouple sensors are positioned initially at the same locations. It can be anticipated that temperature extrapolation will suffer here, since the material is strongly nonlinear; hence as the extrapolation distance increases the variation is felt. This may be signaled by the fact that the linear theory does not approximate the midpoint thermocouple trace closely enough. The results are presented in Table 5 and are expressed as the percent error in the local temperature values. It should be noted that as the degree of the polynomial approximation increases the results improve. Hence the effect of inaccuracies attributed to the degree of the polynomial approximation is isolated. The last three column groups demonstrate how the error increases with distance for backward extrapolation to the surface. Since linear theory allows for adjustments in the values of the thermal diffusivity, an effective thermal diffusivity is now introduced. By arbitrarily adjusting δ so that it produces a better match in the first numerical value of the midposition thermocouple, an effective thermal diffusivity may be determined. This procedure is called thermal diffusivity iteration, its results are indicated in Table 6. Extrapolation accuracy is improved; however, the distortion due to distance is still present. The computations for both tables indicated that if the extrapolation were carried out over a smaller distance, then better results could be effected. In particular, the results for $x=0.0286$ m are adequate for engineering application. Consequently, thermocouples are utilized which are closer to the surface and the results are indicated in Tables 7 and 8. The extrapolated values are significantly improved. The midposition traces are more closely approximated, thereby serving as a better basis for surface temperature evaluation.

Obviously, it is possible to further improve the extrapolation mechanism by selection of temperature sensors still closer to the surface, since in Table 7, the first thermocouple sensor is relatively far from the surface, i.e., $x_1=0.0190$ m. In Table 9 a summary is shown of the computational results for a material experiencing a 50% decrease in thermal conductivity, i.e., $\beta=-0.5$. The temperature sensors are located at $x_1=0.009525$ m and $x_2=0.01905$ m. Comparison of the midvalues indicated that the additional step of thermal diffusivity iteration was not warranted here; consequently, no adjustments were required, $\delta=\delta_i$.

From the two numerical examples presented thus far, it appears that good temperature prediction may be achieved by regarding the original nonlinear situation as linear. In order to achieve this, a zone close to the surface must be identified where the thermal behavior appears to be linear. In the inverse problem this is not difficult to do since many thermocouples are customarily embedded; consequently, the pair of sensors that comes closest to linear behavior is selected to the extrapolation procedure.

In conclusion, two mechanisms for obtaining solutions to the difficult problem of nonlinear heat conduction are developed. The first, the direct case, presents a new method of equivalent linearization incorporating an iteration feature. Furthermore, since the current method does not depend upon the a priori requirement of periodicity, it can therefore be applied to nonlinear parabolic equations with a broader range of companion linear boundary conditions. In particular, as a demonstration of the method's accuracy, two numerical examples are reported. Both examples treat the semi-infinite solid with variable thermal conductivity subject to boundary conditions of the type $T(0,t)=\text{constant}$ and $T(0,t)=B_0\sqrt{t}$. The theoretical development suggests that extensions to other types of linear boundary conditions are feasible.

Though the inverse case is regarded as more formidable, it does have the advantage that temperature extrapolation is not required to encompass "all" space and "all" time. In this second case, concepts of weak and strong nonlinear conditions are introduced. For materials exhibiting weak nonlinear behavior, direct application of linear inverse theory produces results which are excellent. Strong nonlinear materials can be expected to be more difficult to treat. As the extrapolation distance increases, the error in the linear temperature prediction process increases due to the variation in the thermal conductivity. However, this effect can be attenuated by judicious positioning of the temperature sensors close to the surface. Furthermore, the region of interest may also be approximated by a linear material with an effective or equivalent thermal diffusivity. In other words, the original nonlinear region may be replaced by an equivalent one which responds as if it were linear. A numerical example has been presented which demonstrates that the results thus obtained are sufficiently accurate for engineering purposes.

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References

[1] Storm, M.L., "Heat Conduction in Simple Metals," Ph.D. Dissertation, New York University, New York, N.Y., 1951.

[2] Storm, M.L., "Heat Conduction in Simple Metals," *Journal of Applied Physics*, Vol. 22, July 1951, pp. 940-951.

[3] Knight, J.H. and Phillip, J.R., "Exact Solutions in Nonlinear Diffusion," *Journal of Engineering Mathematics*, Vol. 8, July 1974, pp. 219-227.

[4] Pattle, R.E., "Diffusion for an Instantaneous Point Source With a Concentration-Dependent Coefficient," *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. 12, Pt. 4, 1959, pp. 407-409.

[5] Hopkins, M.R., "Heat Conduction in a Medium Having Thermal Properties Depending on the Temperature," *Proceedings of the Physical Society, London,* Vol. 50, 1938, pp. 703-706.

[6] Andre-Talamon, R., "Sur La Diffusion Non-Lineaire De La Chaleur," *International Journal of Heat and Mass Transfer,* Vol. 11, 1968, pp. 1351-1357.

[7] Tsang, T., "Transient State Heat Transfer and Diffusion Problems," *Industrial and Engineering Chemistry,* Vol. 52, Aug. 1960, pp. 707-710.

[8] Clauson, W.W., "On a Nonlinear Problem in Heat Conduction," *Chemical Engineering Progress Symposium Series—Boston,* Vol. 61, Aug. 1963, pp. 16-25.

[9] Olsson, U., "Application of a Perturbation Method to Heat Flow Analysis in Materials Having Temperature-Dependent Properties," *AIAA Journal,* Vol. 8, Oct. 1970, pp. 1902-1903.

[10] Goodman, T.R., "Application of Integral Methods to Transient Nonlinear Heat Transfer," *Advances in Heat Transfer,* Vol. 1, 1964, pp. 51-122.

[11] Imber, M. and Huang, P.N.S., "Phase Change in a Semi-Infinite Solid with Temperature Dependent Thermal Properties, *International Journal of Heat and Mass Transfer,* Vol. 16, 1973, pp. 1951-1954.

[12] Yang, K.T. and Szewczyk, A., "An Approximate Treatment of Unsteady Heat Conduction in Semi-Infinite Solids with Variable Thermal Properties," *Journal of Heat Transfer,* Vol. 26, Aug. 1959, pp. 251-252.

[13] Yang, K.T., "Calculation of Unsteady Heat Conduction in Single Layer and Composite Finite Slabs With and Without Property Variations by an Improved Integral Procedure,"*International Developments in Heat Transfer ASME,* 1963, pp. 18-27.

[14] Chung, B.T.F. and Yeh, L.T., "Nonlinear Transient Heat Transfer in Composite Slabs," ASME Paper 76-Ht-39.

[15] Sugiyama, S., Nishimura, M., and Watanabe, H., "Transient Reponse of Composite Slabs," *International Journal of Heat and Mass Transfer,* Vol. 17, 1974, pp. 875-883.

[16] Yang, K.T., "Transient Conduction in a Semi-Infinite Solid With Variable Thermal Conductivity," *Journal of Applied Mechanics,* Vol. 25, March 1958, pp. 146-147.

[17] Kadambi, V., "Evaluation of Unsteady State Conduction Heat Transfer by a Non-Iterative Finite Difference Method," ASME Paper 76-Ht-72.

[18] Imber, M. and Khan, J., "Prediction of Transient Temperature Distributions with Embedded Thermocouples," *AIAA Journal,* Vol. 10, June 1972, pp. 784-789.

[19] Imber, M., "A Temperature Extrapolation Method for Hollow Cylinders," *AIAA Journal,* Vol. 11, Jan. 1973, pp. 117-118.

[20] D'Souza, N., "Numerical Solution of One-Dimensional Inverse Transient Heat Conduction by Finite Difference Method," ASME Paper 75-WA/HT-81.

[21] Arledge, R.G. and Haji-Sheikh, A., "An Iterative Approach to the Solution of Inverse Heat Conduction Problems," ASME Paper 77-WA/TM-2.

[22] Hore, P.S., Krutz, G.W., and Schoenhals, R.J., "Application of the Finite Element Method to the Inverse Heat Conduction Problem," ASME Paper 55-WA/TM-4.

[23] Chawla, T.C., Leaf, G., Chen, W.L., and Grolmes, M.A., "The Application of the Collocation Method Using Hermite Cubic Splines to Nonlinear Transient One-Dimensional Heat Conduction Problems," *Journal of Heat Transfer,* Vol. 97, Nov. 1975, pp. 562-569.

[24] Beck, J.V., "Nonlinear Estimation Applied to the Nonlinear Inverse Heat Conduction Problem," *International Journal of Heat and Mass Transfer,* Vol. 13, 1970, pp. 703-716.

[25] Mehta, R.C., "Solution of the Inverse Conduction Problem," *AIAA Journal,* Vol. 15, June 1977, p. 1355.

[26] Chen, C.J. and Thomsen, D.M., "On Transient Cylindrical Surface Heat Flux Predicted from Interior Temperature Response," *AIAA Journal,* Vol. 13, May 1975, pp. 697-699.

[27] Imber, M., "Temperature Extrapolation Mechanism for Two-Dimensional Heat Flow," *AIAA Journal,* Vol. 12, Aug. 1974, pp. 1089-1093.

[28] Imber, M., "Two-Dimensional Inverse Conduction Problem—Further Observations," *AIAA Journal,* Vol. 13, Jan. 1975, pp. 114-115.

[29] Vujanovic, B., "Application of the Optimal Linearization Method to the Heat Transfer Problem," *International Journal of Heat and Mass Transfer,* Vol. 16, 1973, pp. 1111-1117.

[30] Iwan, W.D., "On Defining Equivalent Systems for Certain Ordinary Non-Linear Differential Equations," *Journal of Nonlinear Mechanics,* Vol. 4, 1969, pp. 325-334.

[31] Iwan, W.D., "Application of an Equivalent Nonlinear System Approach to Dissipative Dynamical Systems," *Journal of Applied Mechanics,* Vol. 36, Sept. 1969, pp. 412-416.